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Real versus complex β–deformation of the $\mathcal{N} = 4$ planar super Yang-Mills theory

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Abstract

This is a sequel of our paper hep-th/0606125 in which we have studied the $\mathcal{N} = 1$ $SU(N)$ SYM theory obtained as a marginal deformation of the $\mathcal{N} = 4$ theory, with a complex deformation parameter β and in the planar limit. There we have addressed the issue of conformal invariance imposing the theory to be finite and we have found that finiteness requires reality of the deformation parameter β.
In this paper we relax the finiteness request and look for a theory that in the planar limit has vanishing beta functions. We perform explicit calculations up to five loop order: we find that the conditions of beta function vanishing can be achieved with a complex deformation parameter, but the theory is not finite and the result depends on the arbitrary choice of the subtraction procedure. Therefore, while the finiteness condition leads to a scheme independent result, so that the conformal invariant theory with a real deformation is physically well defined, the condition of vanishing beta function leads to a result which is scheme dependent and therefore of unclear significance.
In order to show that these findings are not an artefact of dimensional regularization, we confirm our results within the differential renormalization approach.

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1 Introduction

Recently we have studied marginal deformations of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory best known as β–deformations. These theories are obtained through the following modification of the $\mathcal{N} = 4$ theory: one enlarges the space of parameters adding to the gauge coupling g two complex couplings h and β. These new parameters enter the chiral superpotential via the substitution

$$ig \ \mathrm{Tr}(\ \Phi_1\Phi_2\Phi_3 - \ \Phi_1\Phi_3\Phi_2\ ) \ \longrightarrow \ ih \ \mathrm{Tr}\left(\ e^{i\pi\beta}\ \Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\ \Phi_1\Phi_3\Phi_2\ \right) \qquad (1)$$

It has been argued that these β-deformed $\mathcal{N} = 1$ theories become conformally invariant if the constants g, h and β satisfy one equation in the space of parameters [1]. Of course it is of interest to find this condition explicitly. For the case of β real and in the planar limit we have shown [2] that to all perturbative orders this equation is simply given by

$$h\bar{h} = g^2 \qquad (2)$$

The corresponding conformal theory represents the exact field theory dual to the Lunin–Maldacena supergravity background [3]. Further confirmation of this result can be found in [4, 5].

In a recent paper [6] we have extended our study to the case of complex β [7]. The analysis was performed in the planar limit, using a perturbative approach, superspace techniques and dimensional regularization. With the aim of addressing the issue of conformal invariance we have investigated the finiteness of the theory. In fact simply imposing the finiteness of the two-point chiral correlators we found that only *real* values of the parameter β are allowed, thus leading to the condition in (2). Being the theory finite, this result is obviously independent of the renormalization scheme adopted throughout the calculation. The corresponding theory is conformal invariant and perfectly well defined.

On one hand this result might be somewhat surprising since the expectation was to find an equation for the parameters, g real and h and β complex, with no additional constraints. On the other hand the request of real β seems to be in agreement with results in the string dual approach where singularities appear in the deformed metric as soon as β acquires an imaginary part [3, 8, 9]. Our findings are also consistent with results concerning the integrability of the theory [10, 11].

In this paper we reexamine the problem imposing less restrictive requirements. Here in order to have a conformal theory we simply ask the gauge beta function and the chiral beta function to vanish. The general strategy we have in mind is to define the theory at its conformal point looking for a surface of renormalization group fixed points in the space of the coupling constants. This amounts to perform a coupling constant reduction by expressing the chiral couplings as functions of the gauge coupling g. This operation has an immediate consequence: we are forced to work perturbatively in powers of g instead of powers of loops. This allows different loop orders to mix and in general the conditions which insure finiteness become different from the conditions for vanishing beta functions. Therefore standard finiteness theorems [12, 13] for the gauge beta function cannot be used.

We perform explicit calculations up to five loops and find that the condition of vanishing beta functions can be accomplished with a complex deformation parameter, but the theory is not finite. Thus we are forced to renormalize the theory and consequently the result is dependent on the arbitrary choice of the subtraction procedure. Of course if we want to recover a result that does not depend on the renormalization scheme we have to impose finiteness and then we go back to a real deformation parameter.

In order to make sure that our findings are independent of the regularization procedure we have adopted, i.e. dimensional regularization [1], we have redone various calculations within the differential renormalization approach and confirmed the results.

It is worthwhile emphasizing that the five-loop calculation of the planar gauge beta function is a highly non trivial exercise. We have accomplished it through the use of improved superspace techniques [14, 13] in conjunction with a lot of ingenuity in the D-algebra manipulations. Our result gives indication that a generalization of the standard finiteness theorems [12, 13] for the gauge beta function holds, i.e. if the matter chiral beta function vanishes up to $\mathcal{O}(g^n)$ then the gauge beta function is guaranteed to vanish up to $\mathcal{O}(g^{n+2})$.

The paper is organized as follows. In Section 2 we describe the general approach and briefly review our previous calculation [6]. In Section 3 we present the evaluation of the chiral and vector beta functions. We explicitly show how the conditions of vanishing beta functions do not give a finite theory and explain how the dependence on the renormalization scheme adopted affects the result. In Section 4 we discuss the calculation within the differential renormalization approach with the use of analytic regularization. Final comments are in our conclusions.

2 The general setting and a brief review of conformal invariance of the β–deformed theory via finiteness

We consider the $\mathcal{N} = 1$ β–deformed action written in terms of the superfield strength $W_\alpha = i\bar{D}^2(e^{-gV} D_\alpha e^{gV})$ and three chiral superfields Φ_i with $i = 1, 2, 3$. With notations as in [15] we have

$$\begin{aligned} S &= \int d^8z \ \mathrm{Tr}\left(e^{-gV}\bar{\Phi}_i e^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6z \ \mathrm{Tr}(W^\alpha W_\alpha) \\ &\quad + ih\int d^6z \ \mathrm{Tr}(\ q\ \Phi_1\Phi_2\Phi_3 - \frac{1}{q}\ \Phi_1\Phi_3\Phi_2\ ) \\ &\quad + i\bar{h}\int d^6\bar{z} \ \mathrm{Tr}(\ \frac{1}{\bar{q}}\ \bar{\Phi}_1\bar{\Phi}_2\bar{\Phi}_3 - \bar{q}\ \bar{\Phi}_1\bar{\Phi}_3\bar{\Phi}_2\ ) \qquad , \qquad q \equiv e^{i\pi\beta} \end{aligned} \tag{3}$$

where h and β are complex couplings and g is the real gauge coupling constant. In the undeformed $\mathcal{N} = 4$ SYM theory one has $h = g$ and $q = 1$. In the present case it is

[1] From our experience dimensional regularization always works but it is often questioned.

convenient to define

$$h_1 \equiv hq \qquad\qquad h_2 \equiv \frac{h}{q} \tag{4}$$

and work with couplings g, h_1 and h_2.

In the spirit of [1] (see also [16]-[21]) the idea is to find a surface of renormalization group fixed points in the space of the coupling constants. To this end one reparametrizes these couplings in terms of the gauge coupling g. In fact, since in the planar limit for each diagram the color factors from chiral vertices is always in terms of the products $h_1^2 \equiv h_1\bar{h}_1$ and $h_2^2 \equiv h_2\bar{h}_2$, we express directly h_1^2 and h_2^2 as power series in the coupling g^2 as follows

$$\begin{aligned} h_1^2 &= a_1 g^2 + a_2 g^4 + a_3 g^6 + \dots \\ h_2^2 &= b_1 g^2 + b_2 g^4 + b_3 g^6 + \dots \end{aligned} \tag{5}$$

The final goal is to study the condition that in the large N limit the couplings have to satisfy in order to guarantee the conformal invariance of the theory for complex values of h and β.

In the large N limit for real values of β, i.e. if $q\bar{q} = 1$, the β-deformed theory becomes exactly conformally invariant if the condition (2) is satisfied [2]. In this case the chiral couplings differ only by a phase from the ones of the $\mathcal{N} = 4$ SYM theory and the planar limits of the two theories are essentially the same.

When $q\bar{q} \neq 1$, in order to isolate the relevant terms and drastically simplify the analysis, it is convenient [22] to study the condition of conformal invariance considering the difference between contributions computed in the β-deformed theory and the corresponding ones in the $\mathcal{N} = 4$ SYM theory (which is finite and with vanishing beta function). The simplification is due essentially to the following facts: when computing the difference between graphs in the β-deformed and in the $\mathcal{N} = 4$ theory we need not consider diagrams that contain only gauge-type vertices since their contributions is the same in the two theories. Instead we concentrate on divergent graphs that contain either only chiral vertices or mixed chiral and gauge vertices. In fact the relevant terms come from the chiral vertices that are actually different in the two theories. Addition of vectors simply modifies the color due to the chiral vertices by the multiplication of g^2 factors which are the same for both theories.

The idea is to proceed perturbatively in superspace. The propagators for the vector and chiral superfields, and the interaction vertices are obtained directly from the action in (3). Supergraphs are evaluated performing the D-algebra in the loops and the corresponding divergent integrals are computed using dimensional regularization in $n = 4 - 2\epsilon$.

In [6] these techniques were used to impose the condition of finiteness on the β-deformed theory and to this end it was sufficient to require finiteness of the two-point chiral correlator. We review the relevant steps of the calculation performed in [6] and refer the reader to that paper for technical details.

At order g^2 we have to consider one-loop divergent diagrams in the β-deformed and in the $\mathcal{N} = 4$ theory and compute the difference. This amounts to the evaluation of chiral

bubbles and gives the following divergent contribution to the chiral propagator

$$\frac{N}{(4\pi)^2}\ \left[h_1^2+h_2^2-2g^2\right]\ \frac{1}{\epsilon} \tag{6}$$

Using the expansions in (5), in order to obtain a finite result we have to impose the condition

$$\mathcal{O}(g^2): \qquad\qquad a_1+b_1-2=0 \tag{7}$$

In fact we have shown [6] that the condition

$$h_1^2+h_2^2=2g^2 \tag{8}$$

ensures conformal invariance up to three loops in the planar limit. For the chiral two-point function the only divergences come from the one-loop bubble and this implies that up to order g^6, we find the following additional requirements

$$\begin{aligned}\mathcal{O}(g^4): &\qquad a_2+b_2=0\\ \mathcal{O}(g^6): &\qquad a_3+b_3=0\end{aligned} \tag{9}$$

When we move up to four loops we can repeat the same reasoning as above. Indeed using the condition in (8) we can show that all the four–loop diagrams that either contain vector lines on chiral bubbles or consist of various arrangements of chiral bubbles are not relevant. We simply need to focus on a new type of chiral divergent structure, the one shown in Fig.1.

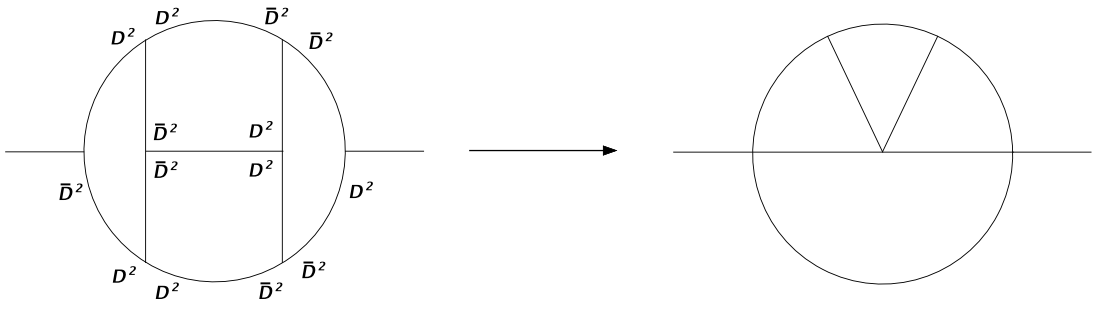

Figure 1: Four-loop supergraph and its associated relevant bosonic integral

From the four-loop calculation in [6] we find that, computing the difference with the corresponding contribution from the $\mathcal{N}=4$ theory and using the expansions (5) finiteness is achieved if

$$\mathcal{O}(g^8): \qquad\qquad a_4+b_4-\frac{5}{2}\ \zeta(5)\ N^3\ \frac{1}{(4\pi)^6}(a_1-b_1)^4=0 \tag{10}$$

For later convenience we define

$$A \equiv \frac{N}{(4\pi)^2}(a_4 + b_4) \qquad\qquad B \equiv -\frac{5}{2}\zeta(5)\frac{N^4}{(4\pi)^8}(a_1 - b_1)^4 \tag{11}$$

so that the previous condition becomes

$$A + B = 0 \tag{12}$$

Then we move to the next order. If we were following a standard procedure, i.e. canceling divergences order by order in loops, having canceled the $1/\epsilon$ pole terms at lower orders we would be guaranteed of the vanishing of the $1/\epsilon^2$ terms at the next order. In our case, instead, we have imposed the finiteness condition order by order in g^2. At the order g^8 this has led us to the relation (12) which allowed us to cancel the $1/\epsilon$ pole from the one-loop diagram with the $1/\epsilon$ pole from the four-loop diagram. When computing the chiral two-point function, these one- and four-loop structures show up at order g^{10} as subdivergences in two-loop and five-loop integrals respectively. It is easy to realize that they produce $1/\epsilon^2$-pole terms. In [6] we have shown that in order to cancel the $1/\epsilon^2$ terms one has to impose $A = B = 0$, i.e.

$$a_1 = b_1 = 1 \qquad \text{and} \qquad a_4 + b_4 = 0 \tag{13}$$

We note that at the order g^8 the finiteness condition (12) is not sufficient to insure the vanishing of the chiral beta function which turns out to be proportional to $A + 4B$ (see eq.(18) in the next Section). Therefore at this order the theory is finite *but* the beta functions do not vanish. However if we take into account the finiteness condition from the order g^{10} we end with $A = B = 0$ which leads to a theory finite and at a RG fixed point.

Under the conditions in (13) $1/\epsilon$ divergences at five and two loops are automatically canceled. Thus at order g^{10} the only divergence in the chiral propagator comes again from the one-loop bubble eq.(6) and we are forced to impose

$$a_5 + b_5 = 0 \tag{14}$$

In [6] we have shown that new chiral graphs at six loops and higher are not relevant. Therefore, everything is controlled by the cancellation of $1/\epsilon$ divergences at one and four loops and of $1/\epsilon^2$ poles at two and five loops. These patterns repeat themselves at the order $(g^2)^{4k}$ and at the order $(g^2)^{4k+1}$ respectively.

The final solution is simply (see [6] for details)

$$a_1 = b_1 = 1 \qquad\qquad a_n = b_n = 0 \qquad n = 2, 3, \ldots \tag{15}$$

which implies $\sinh(2\pi \mathrm{Im}\beta) \sim (h_1^2 - h_2^2) = 0$. Therefore, the β-deformed SYM theory is finite only for β real and, as already stressed, the beta functions also vanish.

We emphasize that this result is independent of the renormalization scheme: had we done the calculation using a different scheme the condition of finiteness would have led us to the solution β real.

In the next section we will relax the finiteness requirement. We want to find the condition that the couplings have to satisfy in the large N limit in order to guarantee the vanishing of the chiral and gauge beta functions. We will find that in this case complex values of β are allowed but the resulting conformal invariant theory depends on the renormalization scheme.

3 Conformal invariance of the β–deformed theory via vanishing of the chiral and gauge beta functions

Now we go back to the action in (3) and compute perturbatively in the large N limit the chiral and gauge beta functions. The request of vanishing beta functions will identify a conformal field theory.
First we consider the chiral beta function β_h. It is well-known that in minimal subtraction scheme β_h is proportional to the anomalous dimension γ of the elementary fields and the condition $\beta_h = 0$ can be conveniently traded with $\gamma = 0$. In our case, even working in a generic scheme, one can easily convince oneself that at a given order in g^2 the proportionality relation between β_h and γ gets affected only by terms proportional to lower order contributions to γ. Therefore, if we set $\gamma = 0$ order by order in the coupling, we are guaranteed that β_h vanishes as well.

Thus we impose the vanishing of γ which we obtain from the computation of the two-point chiral correlator. Up to three loops nothing new happens: the condition in (8) insures the vanishing of γ up to the order g^6 and correspondingly also β_h is zero. Moreover up to this order we can use the results in [13] and we are guaranteed that also the gauge beta function β_g is zero up to the order g^9. This is easily understood since in spite of the redefinition in (5) the request of vanishing anomalous dimensions up to order g^6 works order by order in the loop expansion so that general finiteness theorems [12, 13] hold. At this stage the coefficients in (5) have to satisfy

$$\begin{aligned} &\mathcal{O}(g^2): && a_1 + b_1 - 2 = 0 \\ &\mathcal{O}(g^4): && a_2 + b_2 = 0 \\ &\mathcal{O}(g^6): && a_3 + b_3 = 0 \end{aligned} \tag{16}$$

Things become more subtle at $\mathcal{O}(g^8)$: here the only way to achieve the vanishing of γ is to mix contributions from one loop with contributions from four loops. Repeating the calculation of the divergent integrals, the result is proportional to

$$\frac{1}{\epsilon}\left[A\left(\frac{\mu^2}{p^2}\right)^{\epsilon} + B\left(\frac{\mu^2}{p^2}\right)^{4\epsilon}\right] \tag{17}$$

where A and B were defined in (11) and we have explicitly indicated the factors coming from dimensionally regulated integrals at one and four loops (here p is the external momentum and μ is the standard renormalization mass). The anomalous dimension is given

directly by the finite *log* term in (17) and then we see that at order g^8 the vanishing of the anomalous dimension γ requires

$$\mathcal{O}(g^8): \qquad A + 4B = 0 \tag{18}$$

We emphasize that at this order this condition ensures the vanishing of γ and β_h , but as it appears in (17) the theory is *not finite*. We will come back to this point and discuss its implications below. First we want to show that the condition in (18) is sufficient to insure that β_g is zero up to the order g^{11}.

Contributions to the gauge beta function at $\mathcal{O}(g^{11})$ come from two- and five-loop diagrams. Using standard superspace methods the two-loop calculation is straightforward, but at five loops the number of diagrams involved is large and the calculation looks rather repulsive.[2]

In fact using the background field method and covariant supergraph techniques we are able to perform this high loop calculation exactly. We take advantage of the results obtained in [13] where the structure of higher-loop ultraviolet divergences in SYM theories was analyzed using the superspace background field method and supergraph covariant D-algebra [14]. Using this approach contributions to the effective action beyond one loop can be written in terms of the vector connection Γ_a and the field strengths W_α, $\bar{W}_{\dot{\alpha}}$, but not of the spinor connection Γ_α. This result allows to draw strong conclusions on the structure of UV divergences in SYM theories. It was shown [13] that in regularization by dimensional reduction UV divergent terms can be obtained by computing a special subset of all possible supergraphs. The reasoning can be summarized as follows: at any loop order (with the exception of one loop), after subtraction of UV and IR divergences, the infinite part of contributions to the effective action is local and gauge invariant. By superspace power counting and gauge invariance it must have the form

$$\Gamma_\infty = z(\epsilon)\ \mathrm{Tr} \int d^4x\ d^4\theta\ \Gamma^a \Gamma_b (\delta_a{}^b - \hat{\delta}_a{}^b) \tag{19}$$

where Γ^a is the vector connection from the expansion of the covariant derivatives, i.e. $\nabla_a = \partial_a - i\Gamma_a$, produced in the course of the D-algebra. $z(\epsilon)$ is a singular factor from momentum integration of divergent supergraphs and the n-dimensional $\hat{\delta}_a{}^b$ is produced from symmetric integration. Using the rules of dimensional reduction and the Bianchi identities in terms of covariant derivatives one can show that

$$\mathrm{Tr} \int d^4x\ d^4\theta\ \Gamma^a \Gamma_b (\delta_a{}^b - \hat{\delta}_a{}^b) = -\epsilon\ \mathrm{Tr} \int d^4x\ d^2\theta\ W^\alpha W_\alpha \tag{20}$$

From the above relation it is clear that in order to obtain a divergence the coefficient $z(\epsilon)$ in (19) must contain at least a $1/\epsilon^2$ pole. Moreover the complete result can be obtained by calculating tadpole-type contributions with a $\Gamma^a \Gamma_b \delta_a{}^b$ vertex and then covariantizing them

[2]We recall that in [23] a calculation of similar difficulty was attempted: the four-loop gauge beta function including nonplanar graphs. In that case the relevant coefficient was obtained by an indirect assumption because a direct calculation was too involved.

by the substitution $\delta_a{}^b \to \delta_a{}^b - \hat{\delta}_a{}^b$. Thanks to all of this even the five loop computation becomes manageable.

We describe here the main steps that apply both to the two-loop and to the five-loop calculation. As emphasized above we need consider graphs with internal chiral lines only. Thus, according to the rules in [14], at a given order in loop one draws vacuum diagrams with chiral covariant propagators and ∇^2, $\bar{\nabla}^2$ factors at the chiral vertices. Now, in order to reduce as much as possible the number of terms produced in the course of the ∇-algebra, we do not perform the covariant ∇-integration at this stage but modify the procedure as follows. We want to single out tadpole-type contributions proportional to $\Gamma^a\Gamma_a$, therefore we have to figure out which are the potential sources of such terms. The explicit representation of the chiral covariant propagators is given by

$$\Box_+ = \frac{1}{2}\,\nabla^a\nabla_a - iW^\alpha\nabla_\alpha - \frac{i}{2}(\nabla^\alpha W_\alpha) \qquad\qquad \Box_- = \frac{1}{2}\,\nabla^a\nabla_a - i\bar{W}^{\dot\alpha}\bar{\nabla}_{\dot\alpha} - \frac{i}{2}(\bar{\nabla}^{\dot\alpha}\bar{W}_{\dot\alpha}) \quad (21)$$

Therefore in the expressions above we can disregard the terms involving the field strengths since they do not enter the structure in (19). The $\Gamma^a\Gamma_a$ terms can arise only from the expansion of the covariant operator $\nabla^a\nabla_a$ or from contracted covariant derivatives $\nabla^a \ldots \nabla_a$ produced while performing the ∇-algebra. The net result is that we can immediately expand the covariant propagators as follows

$$\frac{1}{\Box_\pm} \quad \to \quad \frac{1}{\frac{1}{2}\nabla^a\nabla_a} \quad \to \quad \frac{1}{\Box} \;+\; \frac{1}{2}\,\frac{1}{\Box}\Gamma^a\Gamma_a\frac{1}{\Box} \qquad (22)$$

where $\Box = \frac{1}{2}\,\partial^a\partial_a$ is the flat propagator. All the rest we drop since it will not contribute to the structure we are looking for. In this way we obtain two types of diagrams:
I. the ones with flat D^2 and $\bar{D}^2$ factors at the vertices, flat propagators and one $\Gamma^a\Gamma_a$ insertion, for which now standard D-algebra can be performed
and
II. the vacuum diagrams with flat propagators but ∇^2, $\bar{\nabla}^2$ factors at the chiral vertices in which the $\Gamma^a\Gamma_a$ vertex will have to appear after completion of the ∇-algebra.
The relevant terms will be the ones that after subtraction of ultraviolet and infrared subdivergences give rise to $1/\epsilon^2$ contributions.

At the two-loop level the analysis is very simple: the vacuum diagram to be considered is shown in Fig.2a. Following the procedure described above, it is straightforward to realize that only I-type diagrams can give rise to $1/\epsilon^2$ poles and so the calculation reduces to the one presented in [14].

We briefly summarize it here. Expanding the covariant propagators as in (22) one obtains three times the diagram in Fig.3 which corresponds to the term

$$\frac{1}{2}\;\mathrm{Tr}\;(\Gamma^a\Gamma_a)\int\frac{d^nk\;d^nq}{(2\pi)^{2n}}\frac{1}{q^2(q+k)^2k^4} \qquad (23)$$

This integral contains a one-loop ultraviolet subdivergence and it is infrared divergent. It is convenient to remove the IR divergence using the R^* subtraction procedure of [24].

Figure 2: Vacuum diagrams: (a) two-loops and (b) five-loops contributions

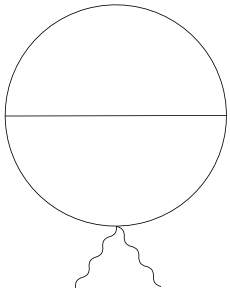

Figure 3: Bosonic two-loop integral

After UV and IR subtraction one isolates the $1/\epsilon^2$ term and rewrites the result in a covariant form. Using (20) it can be recast in the standard divergent part of the two-loop effective action giving a total contribution

$$\frac{N}{(4\pi)^2}\ \frac{3}{4}A\ \frac{1}{\epsilon}\ \mathrm{Tr}\int d^4x\ d^2\theta\ W^\alpha W_\alpha \tag{24}$$

where we have reinserted the A factor defined in (11).

Now we are ready to attack the five-loop calculation which amounts to start with the vacuum diagram in Fig.2b. First we consider the I-type diagrams. In this case expanding the covariant propagators as in (22) we produce twelve times the diagram in Fig.4. We perform standard D-algebra in the loops and look for a contribution that after subtraction of IR and UV subdivergences gives rise to a $1/\epsilon^2$ divergent term. One easily obtains a single contribution corresponding to the bosonic integral shown in Fig.4

$$\frac{1}{2}\ \mathrm{Tr}\ (\Gamma^a\Gamma_a)\int \frac{d^nk\ d^nq\ d^nr\ d^ns\ d^nt}{(2\pi)^{5n}}\frac{1}{r^2(r+q)^2s^2(s+q)^2t^2(t+r)^2(t+s)^2(q+k)^2k^4} \tag{25}$$

The IR divergence is treated as before via R^* subtraction [24] so that, inserting all the factors, the final result is given by

$$\frac{N}{(4\pi)^2}\ \frac{6}{5}B\ \frac{1}{\epsilon}\ \mathrm{Tr}\int d^4x\ d^2\theta\ W^\alpha W_\alpha \tag{26}$$

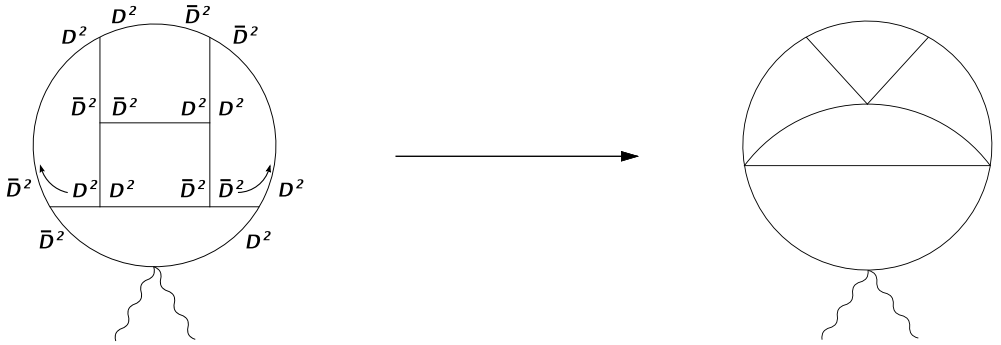

Figure 4: Five-loop supergraph and its associated relevant bosonic integral

with B defined in (11).

In the class of II-type diagrams we have to analyze the vacuum diagram in Fig.5. We operate directly with the covariant spinor derivatives, pushing them through the propagators. Unlike in ordinary D-algebra, covariant spinor derivatives and space-time derivatives contained in the propagators do not commute but it is easy to realize that they generate field strength factors which are not interesting for our calculation. Thus we can commute the ∇_α's through the $\Box^{-1}$. The relevant contributions arise when we produce terms like

$$\nabla^2\bar\nabla^2\nabla^2 = \Box_-\nabla^2 \quad\to\quad -\frac{1}{2}\,\Gamma^a\Gamma_a\nabla^2 \qquad\qquad \bar\nabla^2\nabla^2\bar\nabla^2 = \Box_+\bar\nabla^2 \quad\to\quad -\frac{1}{2}\,\Gamma^a\Gamma_a\bar\nabla^2$$

$$\nabla_\alpha\bar\nabla_{\dot\alpha}\nabla^2 = i\nabla_a\nabla^2 \quad\to\quad \Gamma_a\nabla^2 \qquad\qquad \bar\nabla_{\dot\alpha}\nabla_\alpha\bar\nabla^2 = i\nabla_a\bar\nabla^2 \quad\to\quad \Gamma_a\bar\nabla^2 \tag{27}$$

After all these preliminary observations, now one has to perform the covariant ∇-algebra explicitly and isolate the diagrams that could produce $1/\epsilon^2$ ultraviolet divergences. It turns out that some cleverness must be used in order to reduce the number of the resulting contributions. We show in Fig.5 the successive manipulations that we used to obtain the final answer. As indicated in the figure the first integration by parts of the $\bar\nabla^2$ factor produces three terms: we have denoted by

$$/\!/ \equiv \frac{\frac{1}{2}\nabla^a\nabla_a}{\Box} \;\to\; 1 \;-\; \frac{1}{2}\,\frac{\Gamma^a\Gamma_a}{\Box} \qquad\qquad \blacktriangleright \equiv \nabla_a = \partial_a - i\Gamma_a \tag{28}$$

At this stage we have to work separately on the three graphs and complete the ∇-algebra by disregarding contributions which do not contain $1/\epsilon^2$ divergent terms. (An example of diagram which is not interesting is the one shown in Fig.6. It arises from the second diagram in Fig.5 and would produce only $1/\epsilon$ divergent terms.) In fact if we move the ∇'s judiciously very few relevant terms are generated, the ones schematically shown in Fig.7. Now it is straightforward to show that by integration by parts these potentially relevant graphs do cancel out completely.

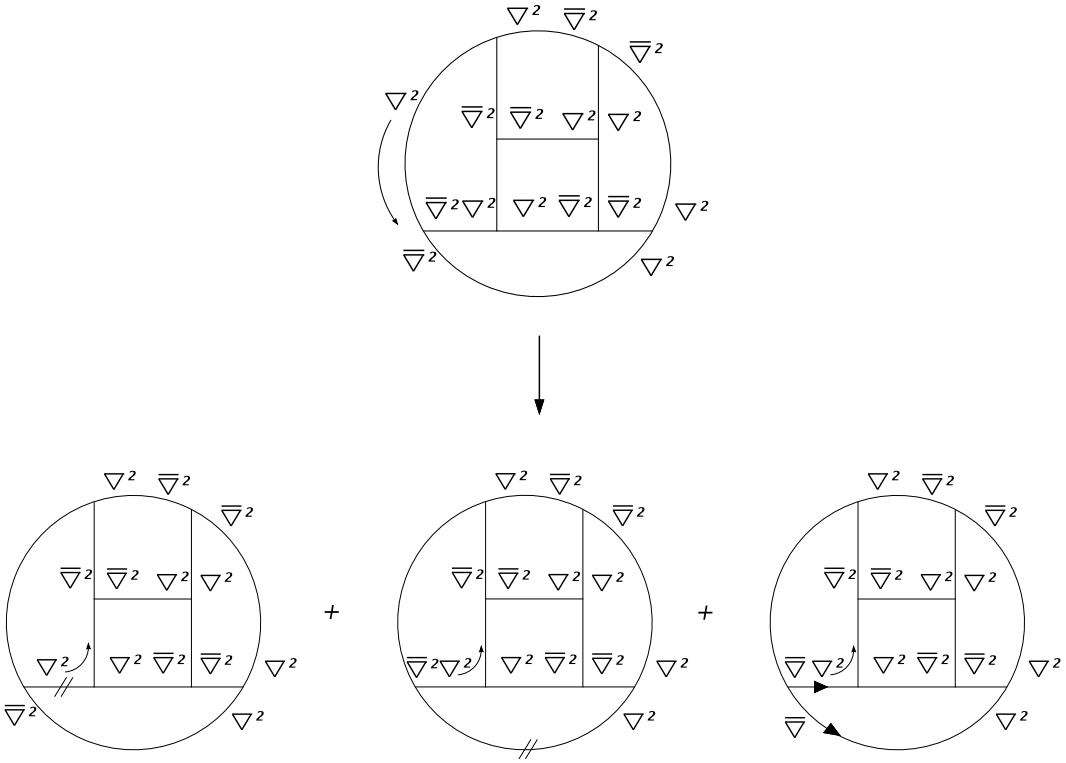

Figure 5: Five-loop vacuum diagram and ∇-algebra operations

In conclusion, the only relevant contributions to the gauge beta function at order g^{11} come from (24) and (26). Using the ordinary prescription to compute beta functions, we find

$$\mathcal{O}(g^{11}): \qquad \beta_g = 0 \qquad \Leftrightarrow \qquad A + 4B = 0 \tag{29}$$

Therefore a single condition on the A and B coefficients is sufficient to define the theory at its conformal point up to the order g^8 and to insure that, despite of the non-finiteness of the theory, the gauge beta function vanishes at the next order.

Now we want to come back to the fact that at order g^8 we have found that the theory subject to the condition in (18) is not finite. In order to proceed consistently we need renormalize the theory adding an appropriate counterterm. As it follows from (17) this will be proportional to the divergence in the form

$$g^8 \ (A + B) \ (\frac{1}{\epsilon} + \rho) \tag{30}$$

where ρ is an arbitrary constant linked to the choice of a finite renormalization. It is worth noticing that the results obtained so far are completely independent of the subtraction scheme we have adopted. In fact even for the calculation of β_g at $\mathcal{O}(g^{11})$ the arbitrary parameter ρ does not enter in the evaluation of the coefficient of the $1/\epsilon^2$ poles from which we read β_g. The issue that now we want to address is what happens to the next order.

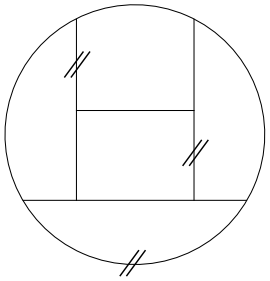

Figure 6: Example of diagram not contributing to the $\frac{1}{\epsilon^2}$ divergence

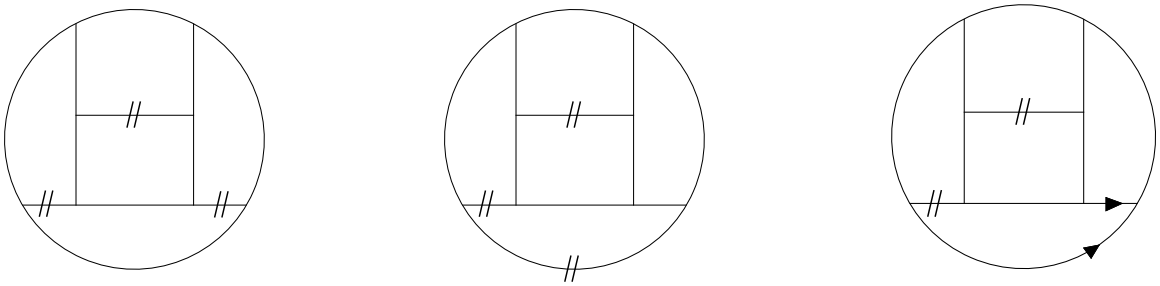

Figure 7: Relevant bosonic integrals associated to the five-loop graph of Fig.5

If we were to push the conformal invariance condition one order higher we should compute the chiral beta function at order g^{10}. We have several sources of nontrivial contributions to γ at this order: one coming from the one-loop bubble proportional to $(a_5 + b_5)$, one from two-loop diagrams and one from five-loop diagrams. In addition we need take into account the contribution from the counterterm in (30) which gives

$$g^{10}\ (A+B)\ (\frac{1}{\epsilon}+\rho)\ \frac{1}{\epsilon}\ \left(\frac{\mu^2}{p^2}\right)^{\epsilon} \tag{31}$$

This last contribution is necessary to appropriately subtract diagrams that contain subdivergences at two and five loops, i.e. the ones that contain $1/\epsilon^2$ poles considered in Section 2. The condition for vanishing γ, obtained as usual from the finite *log* terms, gives an algebraic equation involving A, B and $(a_5 + b_5)$ which, together with (18) allows to determine A and B parametrically and not necessarily vanishing. However the result depends unavoidably on the arbitrary constant ρ which appears in the form

$$(A+B)\ \rho \tag{32}$$

If we wanted to kill the scheme dependence of the result we would need to impose $A+B = 0$ which together with (18) would lead immediately to $A = B = 0$, i.e. the theory is finite and $\mathrm{Im}\beta = 0$.

The comparison of these results with the ones of [6] as summarized in Section 2 leads to the conclusion that the request of conformal invariance via the vanishing of the beta functions is less restrictive than requiring finiteness but the result is scheme dependent.

Pushing the calculations higher we expect to draw the same conclusion: conformal invariance via vanishing beta functions allows for $\mathrm{Im}\beta \neq 0$ but this value and ultimately the conformal theory depend on the choice of the particular renormalization scheme we use.

4 Differential renormalization approach

In order to show that our findings do not depend on the particular regularization used in this Section we reconsider the calculation of the chiral propagator up to the order g^8 in the scheme of differential renormalization.

Differential renormalization works strictly in four dimensions. In its original formulation [25] it is a renormalization without regularization, i.e. it allows for a direct computation of renormalized quantities without the intermediate step of regularizing divergent integrals. In coordinate space the procedure consists in replacing locally singular functions (functions which do not admit a Fourier transform) with suitable distributions defined by differential operators acting on regular functions, where the derivatives have to be understood in distributional sense. The simplest example is the function $1/(x^2)^2$ from the one–loop contribution to $\Gamma^{(2)}$. This function has a non-integrable singularity in $x = 0$. The prescription required by differential renormalization in order to subtract such a singularity is

- We substitute
$$\frac{1}{x^4} \;\rightarrow\; -\frac{1}{4}\Box\frac{\log M^2x^2}{x^2} \tag{33}$$
where M is identified with the mass scale of the theory.

- We understand derivatives in the distributional sense, i.e.
$$\int d^4x f(x)\Box\frac{\log M^2x^2}{x^2} \equiv \int d^4x \Box f(x)\frac{\log M^2x^2}{x^2} \tag{34}$$
for any regular function f.

The two expressions in (33) are identical as long as $x \neq 0$, whereas they differ by a singular term for $x \rightarrow 0$. The substitution (33) can then be understood as adding a suitable counterterm [26]-[28]:

$$\frac{1}{x^4} = -\frac{1}{4}\Box\frac{\log M^2x^2}{x^2} \;+\; c(\alpha)\delta^{(4)}(x) \tag{35}$$

where $c(\alpha)$ can be computed in some regularization scheme and becomes singular when the regularization parameter α is removed.

Having in mind to study conformal invariance and/or finiteness for the deformed theory we need compute both the renormalized chiral propagator and its divergent contributions. As long as we are interested in $\Gamma_R^{(2)}$ we apply the standard differential renormalization prescription (33) order by order in g^2, whereas in order to identify the divergent counterterms which in (33) are automatically subtracted we need introduce a regularization prescription. We compute divergences using the analytic regularization [29].

As noticed above we are interested in computing the difference $(\Gamma^{(2)}_{\rm deformed} - \Gamma^{(2)}_{\mathcal{N}=4})$. Thus at one-loop in coordinate space the contribution from the self-energy diagram is

$$\begin{aligned}\Gamma^{(2)} &= \frac{1}{x^4}\,(h_1^2+h_2^2-2g^2)\frac{N}{(4\pi^2)^2} \qquad (36)\\ &= \frac{1}{x^4}\,\left[(a_1+b_1-2)g^2+(a_2+b_2)g^4+(a_3+b_3)g^6+(a_4+b_4)g^8+\cdots\right]\frac{N}{(4\pi^2)^2}\end{aligned}$$

We renormalize this amplitude by the prescription (33). At order g^2 we find the condition (7) which guarantees finiteness and vanishing of the beta functions.

As already discussed, once the condition (7) is satisfied we can neglect all higher loop diagrams which contain bubbles. In particular, at two and three loops we do not find relevant diagrams. Therefore, at orders g^4 and g^6 only the one-loop expression (36) contributes and the conditions (16) are sufficient to cancel the renormalized and the divergent parts of $1/x^4$.

At order g^8 the pattern changes since besides the contribution from (36) we have the new diagram in Fig.1. After D-algebra, in configuration space it corresponds to

$$-\frac{1}{2}(a_1-b_1)^4 g^8 \frac{N^4}{(4\pi^2)^8}\,\frac{1}{x^2}\int \frac{d^4y\,d^4z\,d^4w}{y^2z^2(y-z)^2(y-w)^2(z-w)^2(x-y)^2(x-w)^2} \qquad (37)$$

This expression has a singularity for $x\sim y\sim z\sim w\sim 0$. To compute its finite part, away from $x=0$ it is convenient to rescale the integration variables as $y\to|x|y$, $z\to|x|z$ and $w\to|x|w$. We are then left with

$$-\frac{1}{2}(a_1-b_1)^4 g^8 \frac{N^4}{(4\pi^2)^8}\,\frac{1}{x^4}\int \frac{d^4y\,d^4z\,d^4w}{y^2z^2(y-z)^2(y-w)^2(z-w)^2(1-y)^2(1-w)^2} \qquad (38)$$

The integral is finite and uniformly convergent for $x\to 0$. It has been computed e.g. in [30] and it gives $20\pi^6\zeta(5)$. At order g^8, summing this contribution to the one-loop result and renormalizing $1/x^4$ as in (33) we obtain

$$\Gamma_R^{(2)}|_{g^8} = (A+4B)\,\left(-\frac{1}{4\pi^2}\Box\frac{\log M^2x^2}{x^2}\right) \qquad (39)$$

where A and B are given in (11).

Therefore, the condition of vanishing γ from $\Gamma_R^{(2)}$ requires $A+4B=0$. This is exactly the condition we have found working in dimensional regularization and momentum space. This is consistent with the fact that the Fourier transform of $\Box\frac{\log M^2x^2}{x^2}$ is $4\pi^2\log p^2/M^2$.

Now we concentrate on the evaluation of the divergent contributions from the one-loop self-energy diagram and from the four-loop diagram in Fig.1. Using analytic regularization in four dimensions, at one loop and order g^8 we have (for simplicity we neglect (2π) factors)

$$A\frac{1}{(x^2)^{2+2l}} \tag{40}$$

whereas at four loops we need evaluate the integral

$$-\frac{N^4}{2}(a_1-b_1)^4 g^8\ \frac{1}{(x^2)^{1+l}}\times \tag{41}$$
$$\int \frac{d^4y\,d^4z\,d^4w}{(y^2)^{1+l}(z^2)^{1+l}[(y-z)^2]^{1+l}[(y-w)^2]^{1+l}[(z-w)^2]^{1+l}[(x-y)^2]^{1+l}[(x-w)^2]^{1+l}}$$

Dimensional analysis allows to compute this integral and obtain $(20\zeta(5)+O(l))\frac{1}{(x^2)^{1+7l}}$. This gives the final answer $4B/(x^2)^{2+8l}$ for the diagram in Fig.1.

Now using the general identity

$$\frac{1}{(x^2)^{2+\alpha l}} \sim \frac{1}{\alpha l}\ \delta^{(4)}(x)\ +\ O(l^0) \tag{42}$$

and summing the one and four-loop results we find that the divergent contribution is

$$A\frac{1}{(x^2)^{2+2l}}+4B\frac{1}{(x^2)^{2+8l}}\ \rightarrow\ (A+B)\ \frac{1}{2l}\ \delta^{(4)}(x) \tag{43}$$

Therefore the cancellation of divergences at order g^8 requires $A+B=0$. If we were to compute the divergences arising at order g^{10} we would find results in total agreement with the results found using dimensional regularization. Going higher in loops we would meet the same pattern an infinite number of times and we would be led to the final result for the coefficients as in (15).

5 Conclusions

We have reexamined the problem of finding superconformal fixed points for β–deformed SYM theories in the large N limit and for the deformation parameter β generically complex. In a previous paper [6] we addressed this issue by requiring the theory to be finite. In this paper instead we have reformulated the problem by requiring the theory to have vanishing beta functions.

Looking for a surface of renormalization fixed points we have expressed the chiral couplings as power expansions in the gauge coupling g (see eq. (5)). This introduces an infinite number of arbitrary coefficients which we fix by requiring order by order either finiteness or zero beta functions.

This coupling constant reduction has an important consequence on the perturbative analysis of the theory. In fact we are forced to work pertubatively in powers of g instead of

powers of loops and at a given order different loops do mix. It follows that the condition of finiteness for the theory at a given order does not necessarily imply that the beta functions vanish and viceversa, in contrast with the case of a standard loop expansion.

Collecting the present results and the ones in [6] the general situation can be then summarized as follows. If we impose the cancellation of UV divergences at a given order we obtain conditions on the coefficients in the expansion (5) which do not set automatically to zero the contribution to the chiral beta function at the same order. In particular, in the planar limit the first nontrivial order where this happens is g^8. However, if we move one order higher and still require the cancellation of divergences we obtain more restrictive conditions on the coefficients and as a by–product all the beta functions at that order vanish. This pattern repeats itself at any order in pertubation theory and leads to the following result: The finiteness condition selects a unique expansion (5) for $h_i(g)$ which corresponds to $\sinh(2\pi \mathrm{Im}\beta) \sim (h_1^2 - h_2^2) = 0$, i.e. to a *real* deformation parameter β.

On the other hand, if we implement superconformal invariance by requiring directly vanishing beta functions regardless of finiteness we obtain less restrictive conditions on the coefficients in (5) and more general solutions $h_i(g)$ to the renormalization group equation $F(g, h_i) = 0$ which defines the surface of fixed points. These solutions correspond in general to theories which are not finite and allow for a *complex* deformation parameter.

In our analysis the term "finiteness" is used in the standard way, that is to indicate a theory which does not have UV divergences at any order in perturbation theory and, consequently, does not require any renormalization. In this sense finiteness is a well-defined and scheme independent statement. Its physical meaning is unquestionable since the set of couplings selected by this condition is uniquely fixed. On the other hand, it is a matter of fact that in the presence of coupling constant reduction the conditions $\beta_h, \beta_g = 0$ turn out to be scheme dependent. This means that the set of couplings which solve these equations is not uniquely determined but depends on the renormalization scheme we chose. In particular, the generically complex value of the deformation parameter β that we find is scheme–dependent. This is the reason why in our approach finiteness and vanishing beta–functions are not equivalent statements.

A more general scenario can be obtained if we relax the request of scheme independence when imposing finiteness. In dimensional regularization scheme dependence can be introduced by hands through the use of evanescent terms [19] in the reduction equations (5). The extra freedom introduced by these ϵ–dependent terms allows to define the theory to be simultaneously finite and at its superconformal point for generically complex *but* scheme dependent β parameters, in agreement with [18, 19, 7]. Therefore, the apparent discrepancy between our results and other statements in the literature [18, 19, 7] can be traced back to the use of a different definition of finiteness.

In the presence of coupling constant reduction we are not guaranteed that finiteness theorems [12, 13] for the gauge beta function are true in their standard version. However, pushing the perturbative calculation up to five loops, we have found that given the chiral beta function vanishing at order g^9, then the gauge beta function is automatically zero at order g^{11}. Our result suggests that the finiteness theorems might be generalized as follows: If the matter chiral beta function vanishes up to the order (g^n) then the gauge

beta function vanishes as well up to the order (g^{n+2}).

We have worked in the planar limit where the condition for superconformal invariance is known exactly [2]. However, the same pattern for finiteness vs. conformal invariance should appear also at finite N. This issue is presently under investigation.

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